Exercise 16

Solve the initial-value problem.

$$y'' + 4y' + 29y = 0, \quad y(0) = 1, \quad y(\pi) = -e^{-2\pi}$$

Solution

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad y' = re^{rx} \quad \to \quad y'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} + 4(re^{rx}) + 29(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 4r + 29 = 0$$

Solve for r.

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(29)}}{2} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$$r = \{-2 - 5i, -2 + 5i\}$$

Two solutions to the ODE are $e^{(-2-5i)x}$ and $e^{(-2+5i)x}$. According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y(x) &= C_1 e^{(-2-5i)x} + C_2 e^{(-2+5i)x} \\ &= C_1 e^{-2x} e^{-5ix} + C_2 e^{-2x} e^{5ix} \\ &= e^{-2x}(C_1 e^{-5ix} + C_2 e^{5ix}) \\ &= e^{-2x}[C_1(\cos 5x - i \sin 5x) + C_2(\cos 5x + i \sin 5x)] \\ &= e^{-2x}[(C_1 + C_2)\cos 5x + (-iC_1 + iC_2)\sin 5x] \\ &= e^{-2x}(C_3 \cos 5x + C_4 \sin 5x) \end{aligned}$$

Apply the boundary conditions to determine C_3 and C_4.

$$y(0) = C_3 = 1$$

$$y(\pi) = e^{-2\pi}(-C_3) = -e^{-2\pi}$$

Solving this system yields $C_3 = 1$ and $C_3 = 1$. Therefore,

$$y(x) = e^{-2x}(\cos 5x + C_4 \sin 5x),$$

where C_4 remains arbitrary.